MATH 2850 HARMONIC MOTION

PART ONE: FREE (UNDAMPED) MOTION

One of the primary applications of second order ODEs is Simple Harmonic Motion. The basic example that we'll be examining is a spring-mass system. Imagine attaching a mass to a spring hanging vertically from a board.

1. The system is said to be in **equilibrium** if the weight of the mass is perfectly balanced against the tension in the spring.

2. The **position** function, denoted $y(t)$, measures the displacement of an object from the equilibrium position t seconds after motion is initiated. $y(t) > 0$ corresponds to the mass being *above* the equilibrium position while $y(t) < 0$ corresponds to the mass being *below* the equilibrium position.

   The standard unit of displacement using Engineering Units is the foot, ft., while the standard unit of displacement in the S.I. system is the meter, m.

3. The **velocity** function, $y'(t)$, is the instantaneous time rate of change of the displacement function. When $y'(t) > 0$ the mass traveling *upwards*, while $y'(t) < 0$ corresponds to the mass traveling *downwards*.

   An object is said to be 'at rest' if $y'(t) = 0$.

4. The **acceleration** function, $y''(t)$, is the instantaneous time rate of change of the velocity function.

5. **Mass**, denoted m, may be defined as a measure of an object's tendency to resist straight line motion.

   The standard unit of mass using Engineering Units is the *slug*, while the standard unit of mass in the S.I. system is the *kilogram*, kg.

6. **Force**, usually denoted F, may be defined as a push or a pull.

   The standard unit of force using Engineering Units is the *pound*, lb., while the standard unit of force in the S.I. system is the *Newton*, N.

   According to Newton's 2nd Law of Motion, Force = Mass * Acceleration.

7. **Weight**, usually denoted w, is the force exerted on an object due to gravity. In accordance to Newton's 2nd Law of motion, Weight = Mass * Acceleration Due to Gravity, or, more compactly: $w = mg$.

   In Engineering Units, $g = 32\,\text{ft/sec}^2$, while in S.I. units, $g = 9.8\,\text{m/sec}^2$. For example, a mass of 2 slugs weighs 64 lbs., while a mass of 10 kg weighs 98 N.

8. **Hooke's Law** states that the restorative force exerted by a spring is directly proportional to the distance the spring is stretched. In symbols, $F = kd$ where k is the spring constant.

EQUATION OF MOTION: Putting all of the above information together, we may derive a differential equation to describe the motion assuming the only forces involved are gravity and the spring's restorative force:

Ans: $my'' + ky = 0$ or $y'' + \dfrac{k}{m}y = 0$. Relabeling $\dfrac{k}{m} = \omega^2$, $y'' + \omega^2 y = 0$.

PROBLEM 1: An 8 lb. weight stretches a spring 6 inches. If this weight is released 1 ft. below the equilibrium position from rest, find the equation of the resulting motion. List the period and frequency of the motion.

How long until the weight passes through the equilibrium position for the first time?

Ans: $y = -\cos(8t)$. Angular frequency: $\omega = 8\,\dfrac{\text{radians}}{\text{second}}$. Ordinary frequency $f = \dfrac{\omega}{2\pi} = \dfrac{4}{\pi}\,\dfrac{\text{cycles}}{\text{second}} = \dfrac{4}{\pi}$ Hz.

Period: $T = \dfrac{2\pi}{\omega} = \dfrac{1}{f} = \dfrac{\pi}{4}$ seconds.

The weight passes through the equilibrium position for the first time at $t = \dfrac{\pi}{16}$ seconds.

PROBLEM 2: A 10 kg mass is attached to a spring. If it is released 1 m above the equilibrium position with a downward velocity of 2 m/sec, the period of the resulting motion is π seconds. Find the equation of motion.

Ans: $y = \cos(2t) - \sin(2t)$

ALTERNATE FORM OF THE SOLUTION: Sinusoid: $y(t) = A\sin(\omega t + \phi)$

Using the sum property of sine, show if $a\sin(\omega t) + b\cos(\omega t) = A\sin(\omega t + \phi)$ then:

$$A^2 = a^2 + b^2, \quad \cos(\phi) = \frac{a}{A}, \quad \sin(\phi) = \frac{b}{A}, \quad \tan(\phi) = \frac{b}{a}$$

PROBLEM 3: Rewrite the solution to PROBLEM 2 as a sinusoid and graph. Check the initial conditions.

$$y = \sqrt{2}\sin\left(2t + \frac{3\pi}{4}\right)$$

PROBLEM 4: Rewrite $y(t) = 2\cos(t) - \sin(t)$ in the alternate form. Qualitatively interpret $y(0)$ and $y'(0)$.

Ans: $y = \sqrt{5}\sin\left(t + \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right)$

The spring was released 2 units above the equilibrium position with a downward velocity of 1.

HOMEWORK: 6.1: Page 278: 3, 5, 7, 9

PART TWO: DAMPED MOTION

In PART ONE, we ignored everything except gravity and the spring force. In this part, we add a term to the differential equation to account for viscous damping - that is, damping which is directly proportional to the instantaneous velocity of the mass. In symbols, we have:

$$my'' = -ky - \beta y',$$

where β is the damping constant ($\beta > 0$). (Note that since the damping force *impedes* the motion, the associated term is given a negative sign.) Rewriting this equation in a more familiar form gives:

$$my'' + \beta y' + ky = 0.$$

To pin down the exact equation of motion, we need only impose suitable initial conditions.

PROBLEM 5: A 2 kg mass is attached to a spring whose spring constant is 16 N/m. A damping force is present which is numerically equal to 12 times the instantaneous velocity. If the mass is released 1 m below the equilibrium position with a downward velocity of 7 m/s, find the equation of motion.

Find the maximum displacement from the equilibrium position.

Does the mass ever return to the equilibrium position?

Ans: $y = \frac{9}{2}e^{-4t} - \frac{11}{2}e^{-2t}$

maximum displacement: $\frac{121}{72}$ meters (below the equilibrium position) when $t = \frac{1}{2}\ln\left(\frac{18}{11}\right)$.

$y(t) < 0$ for $t > 0$ so the mass never returns to the equilibrium position.

PROBLEM 6: The solution to PROBLEM 5 decayed rapidly (being the sum of two exponential functions.) Below are two IVPs where the spring constant has been increased. Solve them and analyze their solutions graphically and analytically.

1. $y'' + 6y' + 9y = 0,\ y(0) = -1,\ y'(0) = -7.$

Ans: $y = -e^{-3t}(1 + 10t)$

2. $y'' + 6y' + 25y = 0,\ y(0) = -1,\ y'(0) = -7.$

   Write your final answer in the form $y = A(t)\sin(\omega t + \phi)$.

Ans: $y = -e^{-3t}\left[\frac{5}{2}\sin(4t) + \cos(4t)\right] = -\frac{\sqrt{29}}{2}\,e^{-3t}\sin\left(4t + \sin^{-1}\left(\frac{2}{\sqrt{29}}\right)\right)$

PROBLEM 7: We may rewrite the differential equation that give rise to damped motion as:

$$y'' + \frac{\beta}{m}y' + \frac{k}{m}y = 0, \quad \text{or} \quad y'' + 2\lambda y' + \omega^2 y = 0.$$

Under what conditions will this differential equation yield oscillating solutions?

Given a system modeled by: $y'' + 2\lambda y' + \omega^2 y = 0$, we say the system is:

- **overdamped** if $\lambda^2 - \omega^2 > 0$.
- **critically damped** if $\lambda^2 - \omega^2 = 0$.
- **underdamped** if $\lambda^2 - \omega^2 < 0$.

PROBLEM 8: If $y(t)$ is an equation for damped motion describe $y(t)$ as $t \to \infty$ in each of the following cases:

- overdamped motion
- critically damped motion
- underdamped motion

HOMEWORK: 6.2: Page 288: 1, 3, 4

PART THREE: FORCED MOTION

In PART TWO, we solved systems whose general equation took the form:

$$my'' + \beta y' + ky = 0.$$

We now introduce a non homogeneous term into this equation in the form of a *forcing* term, $f(t)$:

$$my'' + \beta y' + ky = f(t).$$

We may think of this forcing term as some external input vibrating the support of our spring-mass system, though in other areas (circuits, for instance), this term has special significance (impressed voltage.)

PROBLEM 9: A mass of 1 slug is attached to a spring whose spring constant is 16 lbs./ft. While there is no damping, the system is subject to an external force, $f(t) = 18\sin(5t)$. If the mass is released from its equilibrium position with a downward velocity of 2 ft/sec, find and graph the equation of motion.

Ans: $y = 2\sin(4t) - 2\sin(5t)$

BEATS PHENOMENON: Use the identity below to rewrite your solution to PROBLEM 9:

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Ans: $y = -4\sin\left(\frac{t}{2}\right)\cos\left(\frac{9t}{2}\right)$.

PROBLEM 10: Repeat PROBLEM 9, but instead suppose the external force is given by: $f(t) = 18\sin(4t)$.

Ans: $y = \frac{1}{16}\sin(4t) - \frac{9}{4}t\cos(4t)$

PROBLEM 11: A mass of 1 slug is attached to a spring whose constant is 5 lbs/ft. Initially, the mass is is released 1 ft below the equilibrium with a downward velocity of 5 ft/sec, and the subsequent motion takes place in a medium that offers a damping force numerically equal to 2 times the instantaneous velocity.[1]

1. Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12\cos(2t) + 3\sin(2t)$.
2. Graph the equation of motion along with transient and steady-state solutions on the same coordinate axes.

Ans: $y = 3\sin(2t) - e^{-t}(6\sin(2t) + \cos(2t))$; transient: $-e^{-t}(6\sin(2t) + \cos(2t))$, steady-state: $3\sin(2t)$

HOMEWORK: 6.1: Page 278: 11, 13, 15; 6.2: Page 288: 13, 17

[1] Taken from *A First Course in Differential Equations: The Classic Fifth Edition* by Dennis Zill.

PART FOUR: ANALOGS TO ELECTRIC CIRCUITS - LRC CIRCUITS

Some basic definitions and units:

1. **Charge**, Q is measured in coulombs, c.
2. **Current**, i, is the time rate of change (flow) of charge, $i = Q'(t)$, measured in amperes: $1\,\text{amp} = 1\,\text{c/s}$.
3. **Voltage**, V (or E) is the measure of potential (electromotive force) measured in volts: $1\,\text{V} = 1\,\text{J/c}$.

An LRC circuit has three components: an **inductor** (which produces current), a **resistor** (which impedes current), and a **capacitor** (which stores charge.) The voltage drop across each of these components can be computed by:

- inductor: $V = L\dfrac{di}{dt} = L\dfrac{d^2Q}{dt^2}$, where L is the 'inductance' measured in henrys: $1\,\text{H} = 1\,\text{kg}\cdot\text{m}^2/\text{c}^2$
- resistor: $V = iR = R\dfrac{dQ}{dt}$ where R is the 'resistance' measured in ohms: $1\,\Omega = 1\,\text{J}\cdot\text{s/c}^2$
- capacitor: $V = \dfrac{1}{C}\,Q(t)$ where C is the 'capacitance' measured in farads: $1\,F = 1\,\text{c}^2/\text{J}$

Kirchhoff's Voltage Law is a conservation Law which gives:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}\,Q(t) = E(t),$$

where $E(t)$ is the **impressed voltage** on the circuit.

PROBLEM 12: Do a 'units check' to make sure each term in the DE results in voltage (J/c.)

PROBLEM 13: At $t = 0$, a current of 50 amps flows through an LRC circuit with inductance $L = 0.1\,\mathrm{H}$, resistance $R = 2\,\Omega$, and capacitance $C = 0.01\,\mathrm{F}$. If the initial charge on the capacitor is 2 coulombs and an impressed voltage $E(t) = 650\cos(50t)$ is present, find the current flowing in the circuit.

Identify the transient and steady-state components of your solution.

Ans: $Q(t) = 5e^{-10t}\cos(30t) + 2\sin(50t) - 3\cos(50t)$

$i(t) = Q'(t) = -50e^{-10t}\left(3\sin(30t) + \cos(30t)\right) + 150\sin(50t) + 100\cos(50t)$

transient: $-50e^{-10t}\left(3\sin(30t) + \cos(30t)\right)$; steady-state: $150\sin(50t) + 100\cos(50t)$

HOMEWORK: 6.3: Page: 295: 3, 5, 7, 12*